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Wavefunction of a pair coherent state in the entangled state representation as an eigenfunction of a type of Fokker–Planck differential operator*

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Abstract
We find that the wavefunction of a pair coherent state (or SU(1,1) coherent state) in the entangled state representation is just the eigenfunction of a type of Fokker–Planck differential operator. The relationship between these two states is discussed.

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1. Introduction

It was Einstein–Podolsky–Rosen (EPR) [1] who first used the commutative property of two particles' relative position $X_1 - X_2$ and total momentum $P_1 + P_2$ to initiate the concept of quantum entanglement. Stemming from EPR's idea and using the technique of integration within an ordered product (IWOP) of operators [2, 3], some important entangled state representations of continuum variables are constructed. For example, the common eigenstate of two particles' centre-of-mass coordinate $\frac{1}{2}(X_1 + X_2)$ and the relative momentum $P_1 - P_2$ in two-mode Fock space is [4]

$$|\xi\rangle = \exp\left[-\tfrac{1}{2}|\xi|^2 + \xi a_1^\dagger + \xi^* a_2^\dagger - a_2^\dagger a_1^\dagger\right]|00\rangle, \qquad \xi = \xi_1 + \mathrm{i}\xi_2 = |\xi|\mathrm{e}^{\mathrm{i}\varphi}, \quad (1)$$

which is capable of making up a quantum mechanical representation—the entangled state representation. By using $\langle\xi|$ we have solved some dynamic problems in [5, 6]. In this work, we shall apply $\langle\xi|$ representation to solve the Fokker–Planck-like eigenfunction equation

$$\mathfrak{F}g(\xi, \alpha) = 2\alpha g(\xi, \alpha), \quad (2)$$

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where

$$\mathfrak{F} \equiv \frac{\partial^2}{\partial\xi\partial\xi^*} + \frac{\xi}{2}\frac{\partial}{\partial\xi} + \frac{\xi^*}{2}\frac{\partial}{\partial\xi^*} + \frac{1}{2} + \frac{\xi\xi^*}{4} \tag{3}$$

is a type of Fokker–Planck differential operator, with the eigenfunction $g(\xi, \alpha)$ belonging to the eigenvalue α. Then, we endow this eigenfunction with a definite physical meaning. We may encounter the differential operation in $\mathfrak{F}$ in the study of P-representation of some density operator in the usual coherent state representation [7, 8]; for example, the time evolution of the reduced density operator representing a single-mode electromagnetic field's damping inside a cavity which is the uncorrelated thermal equilibrium mixture of states with the quality factor $\mathfrak{C}$ is governed by the equation [9, 10]

$$\dot{\rho} = -\frac{\mathfrak{C}}{2}\bar{n}_{th}\left(a_1 a_1^\dagger \rho - 2a_1^\dagger \rho a_1 + \rho a_1 a_1^\dagger\right) - \frac{\mathfrak{C}}{2}(\bar{n}_{th}+1)\left(a_1^\dagger a_1 \rho - 2a_1 \rho a_1^\dagger + \rho a_1^\dagger a_1\right), \tag{4}$$

where the first term on the right-hand side describes the transfer of excitations from the nonzero temperature heat bath to the quantum system, while the second term represents the transfer through the decay of photons from the quantum system to the heat bath, $\bar{n}_{th}$ is the mean number of quanta in the thermal reservoir. Letting $|\beta\rangle = \exp\left[-\frac{1}{2}|\beta|^2 + \beta a_1^\dagger\right]|0\rangle$ be the bosonic coherent state [7, 8], using

$$a_1^\dagger|\beta\rangle\langle\beta| = \left(\frac{\partial}{\partial\beta} + \beta^*\right)|\beta\rangle\langle\beta|, \qquad |\beta\rangle\langle\beta|a_1 = \left(\frac{\partial}{\partial\beta^*} + \beta\right)|\beta\rangle\langle\beta|, \tag{5}$$

and deriving the P-representation according to $\rho = \int \frac{d^2\beta}{\pi} P(\beta, \beta^*)|\beta\rangle\langle\beta|$ as Scully and Zubairy did [9], the first part on the right-hand side of (4) naturally appears as

$$a_1 a_1^\dagger|\beta\rangle\langle\beta| - 2a_1^\dagger|\beta\rangle\langle\beta|a_1 + |\beta\rangle\langle\beta|a_1 a_1^\dagger = -\left(2\frac{\partial^2}{\partial\beta\partial\beta^*} + \beta\frac{\partial}{\partial\beta} + \beta^*\frac{\partial}{\partial\beta^*}\right)|\beta\rangle\langle\beta|. \tag{6}$$

By comparing (6) with (3), we see that they have the same differential operation, which is of the Fokker–Planck differential type. Such kind of differential form also appears in the description of a laser's density operator approach, decoherence of quantum oscillator, and some stochastic processes, either via Q function or Wigner function [11]. An interesting question thus arises: how to solve (2) to obtain the eigenfunction $g(\xi, \alpha)$? What is its physical meaning? In the following, we shall show that by combining the entangled state representation and the pair coherent state [12, 13] (or SU(1,1) coherent state [14]) representation we can let the Fokker–Planck differential operation emerge naturally and then show that the solution of (2) is just the wavefunction of the pair coherent state in the entangled state representation. Though the pair coherent state was proposed in 1976 [12], it is scarcely used in mathematical physics. It was ignored for a long time till Agarwal applied it to quantum optics [13]. As Dirac indicated in [15]: 'when one has a particular problem to work out in quantum mechanics, one can minimize the labour by using a representation in which the representatives of the more important abstract quantities occurring in that problem are as simple as possible', we believe that constructing various entangled state representations will be useful not only in treating many problems in quantum optics, but can also open up (explore) new research topics. In the following, we shall show that how the overlap between the entangled state $|\xi\rangle$ and the pair coherent state can lead us to find the solution of equation (2). The work is arranged as follows. In section 2 we briefly review the major features of the entangled state $|\xi\rangle$ and those of the pair coherent state. In section 3 with the use of the eigenvector equation obeyed by the pair coherent state, we set up the eigenfunction equation of a Fokker–Planck operator in the entangled state representation. In section 4 we prove that the eigenfunction of a Fokker–Planck operator is just the wavefunction of the pair coherent state in entangled state representation.

The pair coherent states can be generated by considering the competition of the processes by which photons are either created in pairs or destroyed in pairs. The coherent production or destruction of photons in pairs can take place through the nonlinear mixing process. In the last section we explain why we choose $|\xi\rangle$ representation to examine the feature of a pair coherent state by analysing its classical analogue.

2. A brief review of the features of the entangled state $|\xi\rangle$ and the pair coherent state

In [4] we have shown that the state $|\xi\rangle$ simultaneously obeys the eigenvector equations

$$\left(a_1 + a_2^\dagger\right)|\xi\rangle = \xi|\xi\rangle, \qquad \left(a_1^\dagger + a_2\right)|\xi\rangle = \xi^*|\xi\rangle. \tag{7}$$

Note $\left[a_1 + a_2^\dagger, a_1^\dagger + a_2\right] = 0$, or using

$$X_i = \frac{a_i^\dagger + a_i}{\sqrt{2}}, \qquad P_i = \frac{a_i - a_i^\dagger}{\sqrt{2}i}, \tag{8}$$

we have

$$(X_1 + X_2)|\xi\rangle = \sqrt{2}\xi_1|\xi\rangle, \qquad (P_1 - P_2)|\xi\rangle = \sqrt{2}\xi_2|\xi\rangle. \tag{9}$$

By noting the normal ordering form of the two-mode vacuum state projector

$$|00\rangle\langle 00| =: \mathrm{e}^{-a_1^\dagger a_1 - a_2^\dagger a_2} :, \tag{10}$$

we can immediately prove the completeness relation

$$\int \frac{\mathrm{d}^2\zeta}{\pi}|\xi\rangle\langle\xi| = \int \frac{\mathrm{d}^2\xi}{\pi} : \mathrm{e}^{-|\xi|^2 + (a_1^\dagger + a_2)\xi + \xi^*(a_2^\dagger + a_1) - a_2^\dagger a_1^\dagger - a_1 a_2 - a_1^\dagger a_1 - a_2^\dagger a_2} := 1, \tag{11}$$

and from (7) we can see the orthonormal property

$$\langle\xi|\xi'\rangle = \pi\delta(\xi - \xi')\delta(\xi^* - \xi'^*) \equiv \pi\delta^{(2)}(\xi - \xi'). \tag{12}$$

On the other hand, the pair coherent state in [12, 13] is constructed based on $[Q, a_1 a_2] = 0$, where $Q = a_1^\dagger a_1 - a_2^\dagger a_2$ is the two-mode number-difference operator; $a_1 a_2$ annihilates photons in pair, $\left[a_i, a_j^\dagger\right] = \delta_{ij}$. Assuming q being positive, in the Fock space the pair coherent state is

$$|q, \alpha\rangle = C_q \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{(n+q)!n!}}|n + q, n\rangle, \tag{13}$$

where C_q is the normalization constant:

$$C_q = [(\mathrm{i}|\alpha|)^{-q} J_q(2\mathrm{i}|\alpha|)]^{-1/2}; \tag{14}$$

J_q is the Bessel function. $|q, \alpha\rangle$ is the common eigenvector of the following commuting operators:

$$a_1 a_2|q, \alpha\rangle = \alpha|q, \alpha\rangle, \tag{15}$$

$$Q|q, \alpha\rangle = q|q, \alpha\rangle, \tag{16}$$

$[Q, a_1 a_2] = 0$.

3. Eigenfunction equation of the Fokker–Planck-like operator in the entangled state representation

From (1), we see

$$a_1^\dagger|\xi\rangle = \left(\frac{\partial}{\partial\xi} + \frac{\xi^*}{2}\right)|\xi\rangle, \tag{17}$$

$$a_2^\dagger|\xi\rangle = \left(\frac{\partial}{\partial\xi^*} + \frac{\xi}{2}\right)|\xi\rangle; \tag{18}$$

it then directly follows

$$\langle\xi|a_1a_2 = \left(\frac{\partial}{\partial\xi} + \frac{\xi^*}{2}\right)\left(\frac{\partial}{\partial\xi^*} + \frac{\xi}{2}\right)\langle\xi|, \tag{19}$$

so the Fokker–Planck operator emerges as a representation of the pair annihilators in the entangled state representation. Making overlap of (19) with the pair coherent state vector $|q,\alpha\rangle$ and using (15), we have

$$\langle\xi|a_1a_2|q,\alpha\rangle = \left(\frac{\partial^2}{\partial\xi\partial\xi^*} + \frac{\xi}{2}\frac{\partial}{\partial\xi} + \frac{\xi^*}{2}\frac{\partial}{\partial\xi^*} + \frac{1}{2} + \frac{\xi\xi^*}{4}\right)\langle\xi|q,\alpha\rangle = \alpha\langle\xi|q,\alpha\rangle. \tag{20}$$

Comparing (20) with (2) and (3), we immediately see that the eigensolution $g(\xi,\alpha)$ of the Fokker–Planck operator is just $\langle\xi|q,\alpha\rangle$, the pair coherent state's (or SU(1,1) coherent state) wavefunction in $\langle\xi|$ representation; this is the physical meaning of $g(\xi,\alpha)$. Next, we should calculate $\langle\xi|q,\alpha\rangle$. For this purpose, we use (1), (17) and (18) to examine

$$\begin{aligned} Q|\xi\rangle &= \left(\xi a_1^\dagger - \xi^* a_2^\dagger\right)|\xi\rangle = |\xi|\left(\mathrm{e}^{-\mathrm{i}\varphi}a_1^\dagger - \mathrm{e}^{\mathrm{i}\varphi}a_2^\dagger\right) \\ &\quad\times \exp\left[-\frac{|\xi|^2}{2} + |\xi|\left(\mathrm{e}^{-\mathrm{i}\varphi}a_1^\dagger + \mathrm{e}^{\mathrm{i}\varphi}a_2^\dagger\right) - a_1^\dagger a_2^\dagger\right]|00\rangle = \mathrm{i}\frac{\partial}{\partial\varphi}|\xi\rangle, \end{aligned} \tag{21}$$

then projecting (16) on $\langle\xi|$ representation, we set the equation

$$\langle\xi|Q|q,\alpha\rangle = -\mathrm{i}\frac{\partial}{\partial\varphi}\langle\xi|q,\alpha\rangle = q\langle\xi|q,\alpha\rangle; \tag{22}$$

its solution is

$$\langle\xi|q,\alpha\rangle = B(|\xi|,q,\alpha)\,\mathrm{e}^{\mathrm{i}q\varphi}, \tag{23}$$

where, due to the uniqueness of the wavefunction at φ and $\varphi+2\pi$, $\mathrm{e}^{\mathrm{i}q\varphi}|_{\varphi=0} = \mathrm{e}^{\mathrm{i}q\varphi}|_{\varphi=2\pi}$, q should be integers, and $B(|\xi|,q,\alpha)$ is determined in the following section.

4. Calculating $\langle\xi|q,\alpha\rangle$

Recalling the generating function formula of the two-variable Hermite polynomials $H_{m,n}(\xi,\xi^*)$ [16, 17]

$$\sum_{m,n=0}^{\infty}\frac{t^m t'^n}{m!n!}H_{m,n}(\xi,\xi^*) = \exp(-tt' + t\xi + t'\xi^*), \tag{24}$$

where

$$\begin{aligned} H_{m,n}(\xi,\xi^*) &= \sum_{l=0}^{\min(n,m)}\frac{(-)^l n!m!}{l!(m-l)!(n-l)!}\xi^{m-l}\xi^{*n-l}, \\ &= \frac{\partial^{m+n}}{\partial t^m\partial t'^n}\exp(-tt' + t\xi + t'\xi^*)|_{t=t'=0}; \end{aligned} \tag{25}$$

the entangled state $|\xi\rangle$ can be expanded as

$$|\xi\rangle = \mathrm{e}^{-|\xi|^2/2} \sum_{j,k=0}^{\infty} \frac{1}{\sqrt{j!k!}} H_{j,k}(\xi, \xi^*)|j, k\rangle, \tag{26}$$

where $|j, k\rangle = \frac{(a_1^\dagger)^j (a_2^\dagger)^k}{\sqrt{j!k!}}|0, 0\rangle$ is the two-mode number state. Combining (26) and (13), we derive the overlap

$$\langle\xi|q, \alpha\rangle = C_q \langle\xi| \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!(q+n)!}}|n+q, n\rangle = C_q\, \mathrm{e}^{-|\xi|^2/2} \sum_{n=0}^{\infty} H_{q+n,n}(\xi^*, \xi) \frac{\alpha^n}{n!(q+n)!}. \tag{27}$$

Further, using (25) we see

$$\langle\xi|q, \alpha\rangle = C_q\, \mathrm{e}^{\mathrm{i}q\varphi}\, \mathrm{e}^{-|\xi|^2/2} \sum_{n=0}^{\infty} H_{q+n,n}(|\xi|, |\xi|) \frac{\alpha^n}{n!(q+n)!}. \tag{28}$$

(28) coincides with (23), so

$$B(|\xi|, q, \alpha) = C_q\, \mathrm{e}^{-|\xi|^2/2} \sum_{n=0}^{\infty} H_{q+n,n}(|\xi|, |\xi|) \frac{\alpha^n}{n!(q+n)!}. \tag{29}$$

Comparing $H_{q+n,n}(|\xi|, |\xi|)$ (with the real argument $|\xi|$) with the associated Laguerre polynomials L_n^q [18], we can identify

$$H_{q+n,n}(|\xi|, |\xi|) = n!(-1)^n |\xi|^q L_n^q(|\xi|^2). \tag{30}$$

In order to confirm the correctness of (27), we directly apply the Fokker–Planck-like operator $\mathfrak{F}$ of (3) on $\langle\xi|q, \alpha\rangle$ in (27):

$$\begin{aligned}\mathfrak{F}\langle\xi|q, \alpha\rangle &= C_q \left(\frac{\partial}{\partial\xi^*} + \frac{\xi}{2}\right) \left\{\mathrm{e}^{-|\xi|^2/2} \frac{\partial}{\partial\xi} \sum_{n=0}^{\infty} H_{q+n,n}(\xi, \xi^*) \frac{\alpha^n}{n!(q+n)!}\right\} \\ &= C_q\, \mathrm{e}^{-|\xi|^2/2} \frac{\partial^2}{\partial\xi\partial\xi^*} \sum_{n=0}^{\infty} H_{q+n,n}(\xi, \xi^*) \frac{\alpha^n}{n!(q+n)!}.\end{aligned} \tag{31}$$

By denoting $f = H_{q+n,n}(\xi, \xi^*)$, from (25) we can infer the relations

$$\frac{\partial^2}{\partial\xi\partial\xi^*} f - \xi \frac{\partial}{\partial\xi} f = -(q+n) f, \qquad \frac{\partial^2}{\partial\xi\partial\xi^*} f - \xi^* \frac{\partial}{\partial\xi^*} f = -nf. \tag{32}$$

Therefore, the double-differential operation becomes

$$\begin{aligned}&2\frac{\partial^2}{\partial\xi\partial\xi^*} \sum_{n=0}^{\infty} H_{q+n,n}(\xi, \xi^*) \frac{\alpha^n}{n!(q+n)!} \\ &\quad = \sum_{n=0}^{\infty} \left[\left(\xi \frac{\partial}{\partial\xi} + \xi^* \frac{\partial}{\partial\xi^*}\right) - (q+2n)\right] H_{q+n,n}(\xi, \xi^*) \frac{\alpha^n}{n!(q+n)!}.\end{aligned} \tag{33}$$

Using

$$\frac{\partial}{\partial\xi} H_{q+n,n}(\xi, \xi^*) = (q+n) H_{q+n-1,n}(\xi, \xi^*), \tag{34}$$

$$\frac{\partial}{\partial\xi^*} H_{q+n,n}(\xi, \xi^*) = n H_{q+n,n-1}(\xi, \xi^*), \tag{35}$$

and the recursive relations

$$H_{m+1,n} + nH_{m,n-1} = \xi H_{m,n}, \qquad H_{m,n+1} + mH_{m-1,n} = \xi^* H_{m,n},$$

which can be derived from (25), we see

$$\begin{aligned}
&\left(\xi\frac{\partial}{\partial\xi} + \xi^*\frac{\partial}{\partial\xi^*}\right)\sum_{n=0}^{\infty} H_{q+n,n}(\xi,\xi^*)\frac{\alpha^n}{n!(q+n)!} \\
&\quad = \sum_{n=0}^{\infty}\left[\xi H_{q+n-1,n}(\xi,\xi^*)\frac{\alpha^n}{n!(q+n-1)!} + \xi^* H_{q+n,n-1}(\xi,\xi^*)\frac{\alpha^n}{(n-1)!(q+n)!}\right] \\
&\quad = \sum_{n=0}^{\infty}\left\{[H_{q+n,n}(\xi,\xi^*) + nH_{q+n-1,n-1}(\xi,\xi^*)]\frac{\alpha^n}{n!(q+n-1)!}\right. \\
&\qquad \left. + [H_{q+n,n}(\xi,\xi^*) + (q+n)H_{q+n-1,n-1}(\xi,\xi^*)]\frac{\alpha^n}{(n-1)!(q+n)!}\right\} \\
&\quad = \sum_{n=0}^{\infty}(q+2n)H_{q+n,n}(\xi,\xi^*)\frac{\alpha^n}{n!(q+n)!} + 2\alpha\sum_{n=0}^{\infty}H_{q+n,n}(\xi,\xi^*)\frac{\alpha^n}{n!(q+n)!}.
\end{aligned} \tag{36}$$

Substituting (36) into (33) and then using its result to deal with (31), we obtain

$$\mathfrak{F}\langle\xi|q,\alpha\rangle = 2\alpha C_q\,\mathrm{e}^{-|\xi|^2/2}\sum_{n=0}^{\infty}H_{q+n,n}(\xi,\xi^*)\frac{\alpha^n}{n!(q+n)!} = 2\alpha\langle\xi|q,\alpha\rangle, \tag{37}$$

which coincides with (20). Thus, we have confirmed that the pair coherent state's wavefunction in $\langle\xi|$ representation is just the eigenfunction of the Fokker–Planck-like differential operator; this solution itself seems new. As an application of the result $\langle\xi|q,\alpha\rangle$, we consider the short time evolution of a pair coherent state through the nonlinear mixing process which involves two-photon transition, which is described by an effective Hamiltonian [13]

$$H_{\mathrm{int}} = g\left(a_1^\dagger a_2^\dagger + a_1 a_2\right), \tag{38}$$

where g is the coupling constant depending on the nonlinearity of the medium. Assuming that the initial state is in $|q,\alpha\rangle$, then in a short time Δt, $|q,\alpha\rangle$ will evolve according to the expression

$$\exp(-\mathrm{i}H_{\mathrm{int}}t)|q,\alpha\rangle = (1 - \mathrm{i}H_{\mathrm{int}}t)|q,\alpha\rangle. \tag{39}$$

Then the corresponding wavefunction in $\langle\xi|$ representation is

$$\langle\xi|(1 - \mathrm{i}H_{\mathrm{int}}t)|q,\alpha\rangle = (1 - \mathrm{i}g\alpha\Delta t)\langle\xi|q,\alpha\rangle - \mathrm{i}g\Delta t\langle\xi|a_1^\dagger a_2^\dagger|q,\alpha\rangle, \tag{40}$$

where

$$\langle\xi|a_1^\dagger a_2^\dagger|q,\alpha\rangle = \left(-\xi^*\frac{\partial}{\partial\xi^*} - \xi\frac{\partial}{\partial\xi} - \alpha - 1\right)\langle\xi|q,\alpha\rangle. \tag{41}$$

Substituting (41) into (40) and using (27), we obtain the variation of wavefunction

$$\begin{aligned}
&\langle\xi|(-\mathrm{i}H_{\mathrm{int}}t)|q,\alpha\rangle = \mathrm{i}g\Delta t\left(\xi^*\frac{\partial}{\partial\xi^*} + \xi\frac{\partial}{\partial\xi} + 1\right)\langle\xi|q,\alpha\rangle \\
&\quad = \mathrm{i}g\Delta t C_q\,\mathrm{e}^{-|\xi|^2/2}\sum_{n=0}^{\infty}\left\{(1 - |\xi|^2)H_{q+n,n}(\xi,\xi^*)\frac{\alpha^n}{n!(q+n)!}\right. \\
&\qquad \left. + \xi^* H_{q+n,n-1}(\xi,\xi^*)\frac{\alpha^n}{(n-1)!(q+n)!} + \xi H_{q+n-1,n}(\xi,\xi^*)\frac{\alpha^n}{n!(q+n-1)!}\right\}.
\end{aligned} \tag{42}$$

5. Discussions

(1) Here, we explain why we introduced the entangled state $\langle\xi|$ to examine the feature of the pair coherent state. In the pioneering work of Bhaumik *et al* [12], they noted that the classical analogue of the state $|q,\alpha\rangle$ with definite q is obtained by constraining the two oscillators described by the Hamiltonian

$$H = \frac{p_1^2}{2m} + \frac{1}{2}m\omega^2 x_1^2 + \frac{p_2^2}{2m} + \frac{1}{2}m\omega^2 x_2^2 \tag{43}$$

to oscillate, keeping the difference of their action functions

$$\frac{E_1}{\omega} - \frac{E_2}{\omega} = \text{fixed.} \tag{44}$$

This quantity corresponds to q (since $(E_1 - E_2)/\omega$ corresponds to $a_1^\dagger a_1 - a_2^\dagger a_2$). Bhaumik *et al* then used a generating function [19] to perform a canonical transformation from (x_1, x_2, p_1, p_2) to $(\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{P}_1, \mathfrak{P}_2)$ so that H becomes $\omega(\mathfrak{P}_1 - \mathfrak{P}_2)$. Thus, $\mathfrak{X}_i, \mathfrak{P}_i$ take the role of an (phase) angle and action variable respectively, and $\mathfrak{P}_1 - \mathfrak{P}_2 = \frac{E_1}{\omega} - \frac{E_2}{\omega}$ is canonically conjugate to the relative phase $\frac{1}{2}(\mathfrak{X}_1 - \mathfrak{X}_2)$ of the two oscillators. On the other hand, in equation (21) we have noted that $Q = a_1^\dagger a_1 - a_2^\dagger a_2$ corresponds to $\mathrm{i}\frac{\partial}{\partial\varphi}$ (which is canonically conjugate to angle φ) in $\langle\xi|$ representation which can be viewed as an action variable too, so this naturally led us to employ $|\xi\rangle$ to further examine the new property of the pair coherent state. As one can see from (21) and (23), the $U(1)$ phase $\mathrm{e}^{\mathrm{i}q\varphi}$ appears only in $|\xi\rangle$ representation; this indicates that some important physical property of $|q,\alpha\rangle$ (which is an entangled state too, since its Schmidt decomposition is shown in (13)) can only be seen clearly in the entangled state $|\xi\rangle$ representation.

(2) We now discuss the last term $\frac{\xi\xi^*}{4}$ on the right-hand side of equation (3); this kind of terms, though does not appear in the P-representation of equation (4), may appear in the calculation of the Q function when the master equation (3) is converted into a Fokker–Planck differential equation for the Q function. See the paper of Kim and Bužek (the appendix of [11]) and references therein.

In summary, we have found a new solution to the Fokker–Planck-like eigenfunction equation that is just the wavefunction of a pair coherent state in the entangled state representation. This observation shows that the entangled state representation may have potential uses in solving some Fokker–Planck equations.

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